Generating functions
What were they again?

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Abstract

Two decades ago, I followed some courses on discrete mathematics. One of the topics that has always intriged me was "generating functions"; a polynome that represents a series of numbers.

With this article I try to explain myself why generating functions are useful.

1 Infinite sequences

One of the reasons that generating functions appeal to me is that they have to do with sequences of numbers; even *infinite* sequences of numbers. As an example, let's have a look at such a sequence:

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$$

There are several ways to specify (write down) such a sequence. And some focus on the individual elements, others on the sequence as a whole.

- **Enumeration**
  We could say we have a sequence a, with $a = \langle 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots \rangle$. This stresses the sequence as a whole. Alternatively, we could say $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, $\ldots$, which stresses the individual elements. Enumeration is clear (if it is simple), but not precise, nor concise.

- **Recurrence relation**
  A mathematically more sound way of denoting sequence a is to use a recurrence relation. For this we have to make the judgement that each element in the sequence is half its predecessor, and that we start with 1. So, the recurrence relation is

  $$a_0 = 1$$
  $$a_n = \tfrac{1}{2}a_{n-1}\ \text{, for } n \geq 1$$

- **Closed form**
  The wholy grail of an infinite sequence is finding the *closed form* for a_i. An expression for a_i is in closed form if we can compute it in a fixed number of steps (independent of i), using only "standard" operators.

The closed form for our sequence is $a_i = \frac{1}{2}^i$ (which puts the stress on individual elements), or (putting the stress on the sequence)

$$a = \left\langle \frac{1}{2}^i \right\rangle_{i=0}^{i=\infty}$$

One way to prove that the closed form is indeed equivalent to the recurrence relation is to use induction. For the basis, $n = 0$, the recurrence relation gives us $a_0 = 1$ and the closed form gives us $a_0 = \frac{1}{2}^0 = 1$, so the induction basis is correct. For the "step", $n \geq 1$, we observe that $a_n = \frac{1}{2}a_{n-1} = \frac{1}{2} \times \frac{1}{2}^{n-1} = \frac{1}{2}^n$ (first step due to recurrence relation, second due to induction hypothesis, and third is calculus), which proves the closed form.

- **Generating function**
  The forth notation for a finite sequence is the *generating function*[1], a polynome in an auxiliary variable z. The polynome has a special form: the coefficient of the z^i is a_i:

  $$a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots = \sum_{i=0}^{i=\infty} a_i z^i$$

  which, for our sequence, boils down to

  $$a(z) = 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \ldots = \sum_{i=0}^{i=\infty} \frac{1}{2}^i z^i$$

  Ok, this is possible, but definitely bizar. What's the use of this?

2 Closed form for a generating function

Generating functions are helpful, because they allow us to manipulate the whole sequence as one object. This becomes more apparent, when we are able to condense the generating function to a closed form.

We will use the following infinite sequence, which is a slight generalisation of our previous sequence.

$a_0 = 1$
$a_n = r \times a_{n-1}$, for $n \geq 1$ and some number r

To compute the closed form we proceed as follows

$$a(z)$$

= { definition generating function }

$$\sum_{i=0}^{i=\infty} a_i z^i$$

[1] "Voortbrengende functie" in Dutch

$$
\begin{array}{ll}
= & \quad \{ \text{ split-off the basis } i = 0 \text{ of the recurrence relation } (a_0 = 1) \ \} \\
 & 1 + \sum_{i=1}^{i=\infty} a_i z^i \\
= & \quad \{ \text{ apply “step” of recurrence relation } \} \\
 & 1 + \sum_{i=1}^{i=\infty} r a_{i-1} z^i \\
= & \quad \{ \text{ dummy substitution } i \leftarrow k+1 \ \} \\
 & 1 + \sum_{k=0}^{k=\infty} r a_k z^{k+1} \\
= & \quad \{ \text{ factor out } r \text{ and } z \ \} \\
 & 1 + rz \sum_{k=0}^{k=\infty} a_k z^k \\
= & \quad \{ \text{ definition generating function } \} \\
 & 1 + rza(z)
\end{array}
$$

We conclude that $a(z) = 1 + rza(z)$, or $a(z) - rza(z) = 1$, and thus

$$a(z) = \frac{1}{1 - rz}$$

In other words

$$\text{the generating function } \frac{1}{1-rz} \text{ represents the sequence } \left\langle r^i \right\rangle_{i=0}^{i=\infty} \qquad (\dagger)$$

This is useful, because we can now manipulate (and reason) about the sequence by manipulating (and reasoning) about $\frac{1}{1-rz}$. This is illustrated in the rest of this article.

3 Fibonacci

We have seen that we can represent an infinite sequence by a generating function, and we have seen that we can capture a generating function in closed form. We will now derive a closed form for the *elements* of the Fibonacci sequence. First, we compute a closed form for the *generating function* that represents the Fibonacci sequence.

The Fibonacci sequence f is defined by the following recurrence relation.

$$
\begin{array}{l}
f_0 = 0 \\
f_1 = 1 \\
f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 2
\end{array}
$$

We derive

$$
\begin{array}{ll}
 & f(z) \\
= & \quad \{ \text{ definition generating function } \}
\end{array}
$$

$$
\begin{array}{ll}
 & \sum_{i=0}^{i=\infty} f_i z^i \\
= & \quad \{ \text{ split-off the basis } i = 0 \text{ and } i = 1 \text{ of the recurrence relation } \} \\
 & 0 + z + \sum_{i=2}^{i=\infty} f_i z^i \\
= & \quad \{ \text{ apply "step" of recurrence relation } \} \\
 & z + \sum_{i=2}^{i=\infty} (f_{i-1} + f_{i-2}) z^i \\
= & \quad \{ \text{ split sum } \} \\
 & z + \sum_{i=2}^{i=\infty} f_{i-1} z^i + \sum_{i=2}^{i=\infty} f_{i-2} z^i \\
= & \quad \{ \text{ dummy substitution } i \leftarrow k+1 \text{ and } i \leftarrow l+2 \} \\
 & z + \sum_{k=1}^{k=\infty} f_k z^{k+1} + \sum_{l=0}^{l=\infty} f_l z^{l+2} \\
= & \quad \{ \text{ factor out } z \text{ and } z^2 \} \\
 & z + z \sum_{k=1}^{k=\infty} f_k z^k + z^2 \sum_{l=0}^{l=\infty} f_l z^l \\
= & \quad \{ \text{ definition generating function } \} \\
 & z + z(f(z) - f_0) + z^2 f(z)
\end{array}
$$

So that $f(z) = z + zf(z) + z^2 f(z)$.

In other words, the closed form of the generating function for the Fibonacci sequence is

$$f(z) = \frac{z}{1 - z - z^2}$$

Helas. If we would have found that $f(z) = \frac{1}{1-1.6z}$ we would be ready now, because then we would have concluded that $f_i = 1.6^i$ based on † in Section 2.

Now what?

The trick is to see that we can split the "hard" fraction is two "easy" fractions that do fit our profile. Would it be possible to write $f(z)$ as $\frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$?

$$\frac{A}{1-\alpha z} + \frac{B}{1-\beta z} = \frac{A(1-\beta z) + B(1-\alpha z)}{(1-\alpha z)(1-\beta z)} = \frac{(A+B) - (A\beta + B\alpha)z}{1 - (\alpha+\beta)z + \alpha\beta z^2}$$

This should now equal $\dfrac{z}{1-z-z^2}$, so we should have (equating numerator and denominator)

$$
\begin{array}{l}
(A+B) - (A\beta + B\alpha)z = z \\
1 - (\alpha+\beta)z + \alpha\beta z^2 = 1 - z - z^2
\end{array}
$$

Looking at the denominator, we conclude that $\alpha+\beta = 1$ and $\alpha\beta = -1$. Substituting $\beta = 1-\alpha$ we get $\alpha(1-\alpha) = -1$ or $\alpha^2 - \alpha - 1 = 0$. This gives $\alpha = \frac{1\pm\sqrt{1-4\cdot1\cdot-1}}{2\cdot-1} = \frac{1\pm\sqrt{5}}{2}$. Substituting this back in $\beta = 1-\alpha$ gives $\beta = \frac{2}{2} - \frac{1\pm\sqrt{5}}{2} = \frac{1\mp\sqrt{5}}{2}$. As expected, α and β can be interchanged.

For those who know the *golden ration*, $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, observe that $\alpha = \phi$ and $beta = \hat{\phi}$, where we take the liberty to define $\hat{\phi} = \frac{1}{2} - \frac{1}{2}\sqrt{5}$.

To find values for A and B, we look at the numerator, and conclude $A+B = 0$ and $A\beta + B\alpha = -1$. Substituting $B = -A$ we get $A\beta - A\alpha = -1$ or $A(\alpha - \beta) = 1$ or, since $\alpha - \beta = \sqrt{5}$, $A = 1/\sqrt{5}$ and thus $B = -1/\sqrt{5}$.

Pheff. This was not too easy. But what is the conclusion?

$$f(z) = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z} = \frac{1/\sqrt{5}}{1-\phi z} - \frac{1/\sqrt{5}}{1-\hat{\phi} z}$$

Does this help? Yes!

$$\begin{array}{ll}
 & f(z) \\
= & \quad \{ \text{ above } \} \\
 & 1/\sqrt{5}\frac{1}{1-\phi z} - 1/\sqrt{5}\frac{1}{1-\hat{\phi} z} \\
= & \quad \{ \text{ the } \dagger \text{ rule } \} \\
 & 1/\sqrt{5}\sum_{i=0}^{i=\infty} \phi^i z^i - 1/\sqrt{5}\sum_{i=0}^{i=\infty} \hat{\phi}^i z^i \\
= & \quad \{ \text{ distributing } 1/\sqrt{5} \text{ in both sums } \} \\
 & \sum_{i=0}^{i=\infty} 1/\sqrt{5}\phi^i z^i - \sum_{i=0}^{i=\infty} 1/\sqrt{5}\hat{\phi}^i z^i \\
= & \quad \{ \text{ merging the sums } \} \\
 & \sum_{i=0}^{i=\infty} \left(1/\sqrt{5}\phi^i - 1/\sqrt{5}\hat{\phi}^i\right) z^i
\end{array}$$

Due to † in Section 2, we can now write

$$f_n = 1/\sqrt{5}\phi^n - 1/\sqrt{5}\hat{\phi}^n$$

and this is a closed form for the elements of the Fibonacci sequence.

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